Løsningsforslag R2 eksamen 2015 - Del 1

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November 27, 2015

Oppgave 1

a) $f'(x) = -10\sin(2x)$

b) $g'(x) = \sin x + x\cos x$

c) $h'(x) = 5\left(-e^{-x}\sin(2x) + 2e^{-x}\cos(2x)\right) = 5e^{-x}\left(2\cos(2x) - \sin(2x)\right)$

Oppgave 2

a) $\displaystyle\int_0^2 (x^2 - 2x + 1)\,dx = \left[\frac{1}{3}x^3 - x^2 + x\right]_0^2 = \frac{8}{3} - 4 + 2 = \frac{8-6}{3}$

$= \dfrac{2}{3}$

b) La $u = e^x + 1$, så $du = e^x\,dx$.

$$\therefore \int \frac{e^x}{(e^x+1)^2}\,dx = \int u^{-2}\,du = -u^{-1} + C = C - \frac{1}{e^x+1}, C \in \mathbb{R}$$

Oppgave 3

$$V = \pi\int_0^{\ln 3} (f(x))^2\;dx = \pi\int_0^{\ln 3} 4e^{-x}\,dx = 4\pi\left[-e^{-x}\right]_0^{\ln 3}$$

$$= 4\pi\left(-e^{-\ln 3} + e^{-0}\right) = 4\pi\left(-e^{\ln 3^{-1}} + 1\right) = 4\pi\left(1 - \frac{1}{3}\right) = 4\pi\cdot\frac{2}{3}$$

$= \dfrac{8}{3}\pi$, hvilket skulle vises.

Oppgave 4

a) $F'(4) = f(4) = 1$

b) $A = \int_1^4 f(x)\,dx = F(4) - F(1) = 6 - (-1) = 7$

Oppgave 5

a) $4^2 - 2 \cdot 4 + 1^2 + 6 \cdot 1 + 2^2 - 4 \cdot 2 - 11 = 16 - 8 + 1 + 6 + 4 - 8 - 11$

$= 16 - 16 + 11 - 11 = 0$, så $P(4, 1, 2)$ ligger på kuleflaten.

b) $x^2 - 2x + y^2 + 6y + z^2 - 4z - 11 = 0$

$\therefore (x^2 - 2x + 1) + (y^2 + 6y + 9) + (z^2 - 4z + 4) = 11 + 1 + 9 + 4$

$\therefore (x - 1)^2 + (y + 3)^2 + (z - 2)^2 = 25 = 5^2$, så

Sentrum $= S(1, -3, 2)$, Radius $r = 5$.

c) Normalvektor $\vec{n} = \vec{PS} = [4 - 1, 1 - (-3), 2 - 2] = [3, 4, 0]$, så planet har likningen

$3(x - 4) + 4(y - 1) = 0$

$\therefore 3x + 4y = 16$

Oppgave 6

a) $\sin(2x) = \sin(x + x) = \sin x \cos x + \cos x \sin x = 2 \sin x \cos x$
$\cos(2x) = \cos(x + x) = \cos x \cos x - \sin x \sin x - \cos^2 x - \sin^2 x$

b) $$\begin{aligned}\sin(3x) &= \sin(2x + x)\\ &= \sin(2x)\cos x + \cos(2x)\sin x\\ &= 2\sin x\cos^2 x + \sin x(\cos^2 x - \sin^2 x)\\ &= 3\sin x\cos^2 x - (\sin x)^3\\ &= 3\sin x(1 - \sin^2 x) - (\sin x)^3\\ &= 3\sin x - 4(\sin x)^3, \text{ hvilket skulle vises.}\end{aligned}$$

Oppgave 7

a) $$\begin{aligned}\vec{AB} \times \vec{AC} &= [2-1, -3-2, 4-(-2)] \times [-2-1, 3-2, 1-(-2)]\\ &= [1, -5, 6] \times [-3, 1, 3]\\ &= \begin{vmatrix} \vec{e_1} & \vec{e_2} & \vec{e_3}\\ 1 & -5 & 6\\ -3 & 1 & 3\end{vmatrix}\\ &= [-15-6, -18-3, 1-15]\\ &= [-21, -21, -14]\\ &= -7[3, 3, 2]\end{aligned}$$

b) C ligger på linjen gjennom A og B
$\iff \vec{AB} || \vec{AC}$
$\iff \vec{AB} \times \vec{AC} = \vec{0}$

Ettersom $\vec{AB} \times \vec{AC} = -7[3, 3, 2] \neq \vec{0}$, ligger ikke C på linjen gjennom A og B.

Normalvektor $\vec{n} = -\dfrac{1}{7}\vec{AB} \times \vec{AC} = [3, 3, 2]$, så planet α har likningen

$3(x-1) + 3(y-2) + 2(z-(-2)) = 0$

$\therefore 3x + 3y + 2z = 5$

d) $3 \cdot 2 + 3 \cdot 2 + 2 \cdot 3 = 6 + 6 + 6 = 18 \neq 5$, så D ligger ikke i planet α.

Oppgave 8

$y^2 \cdot y' = x, y(0) = 2$

Metode 1

$\therefore \dfrac{d}{dx}\dfrac{1}{3}y^3 = x + \dfrac{C}{3}, C \in \mathbb{R}$

$\therefore y = \left(\dfrac{3}{2}x^2 + C\right)^{\frac{1}{3}}$

Metode 2

$y^2 \dfrac{dy}{dx} = x$

$\therefore \displaystyle\int y^2\,dy = \int x\,dx$

$\therefore \dfrac{1}{3}y^3 = \dfrac{1}{2}x^2 + C_1, C_1 \in \mathbb{R}$

$\therefore y^3 = \dfrac{3}{2}x^2 + C,\ \text{hvor } C = 3C_1$

$\therefore y = \left(\dfrac{3}{2}x^2 + C\right)^{\frac{1}{3}}$

$y(0) = 2 \Rightarrow C^{\frac{1}{3}} = 2 \Rightarrow C = 2^3 = 8$, så

$y(x) = \left(\dfrac{3}{2}x^2 + 8\right)^{\frac{1}{3}}$

Oppgave 9

Base case: $n = 1$

$VS = 1^3 = 1 = \frac{4}{4} = \frac{2^2}{4} = \frac{1^2(1+1)^2}{4} = HS.$

Induksjon:

Anta at det finnes $n \in \mathbb{N}$ slik at $1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$.

Da har vi at
$$
\begin{aligned}
1^3 + 2^3 + 3^3 + \cdots n^3 + (n+1)^3 &= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\
&= \frac{n^2(n+1)^2 + 4(n+1)^3}{4} \\
&= \frac{(n+1)^2\left(n^2 + 4(n+1)\right)}{4} \\
&= \frac{(n+1)^2(n^2 + 4n + 4)}{4} \\
&= \frac{(n+1)^2(n+2)^2}{4} \\
&= \frac{(n+1)^2((n+1)+1)^2}{4},
\end{aligned}
$$
så påstanden gjelder for $n + 1$

Dermed er påstanden bevist for alle $n \in \mathbb{N}$ ved induksjon.