DEFINITIONS Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$ is**

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

We assume that the 4'th derivative of a function is a constant this allows us to write for any x:

$$\frac{d^4y}{dx^4}(x) = \frac{d^4y}{dx^4}(a)$$

Lets find the third derivative

$$\frac{d^3y}{dx^3}(x) = \frac{d^4y}{dx^4}(a)x + C$$

In order to find C we set x=a

$$C = \frac{d^3y}{dx^3}(a) - \frac{d^4y}{dx^4}(a)a$$

So that

$$\frac{d^3y}{dx^3}(x) = \frac{d^4y}{dx^4}(a)x + \frac{d^3y}{dx^3}(a) - \frac{d^4y}{dx^4}(a)a = \frac{d^4y}{dx^4}(a)(x-a) + \frac{d^3y}{dx^3}(a)$$

Then we find the second derivative

$$\frac{d^2y}{dx^2}(x) = \frac{d^4y}{dx^4}(a)\frac{1}{2}(x-a)^2 + \frac{d^3y}{dx^3}(a)x + C$$

Solve for C:

$$\frac{d^2y}{dx^2}(a) = \frac{d^4y}{dx^4}(a)\frac{1}{2}(a-a)^2 + \frac{d^3y}{dx^3}(a)a + C$$

$$\frac{d^2y}{dx^2}(a) - \frac{d^3y}{dx^3}(a)a = C$$

$$\begin{aligned}\frac{d^2y}{dx^2}(x) &= \frac{d^4y}{dx^4}(a)\frac{1}{2}(x-a)^2 + \frac{d^3y}{dx^3}(a)x + \frac{d^2y}{dx^2}(a) - \frac{d^3y}{dx^3}(a)a \\ &= \frac{d^4y}{dx^4}(a)\frac{1}{2}(x-a)^2 + \frac{d^3y}{dx^3}(a)(x-a) + \frac{d^2y}{dx^2}(a)\end{aligned}$$

Integrating to look for the first derivative

$$\frac{dy}{dx}(x) = \frac{d^4y}{dx^4}(a)\frac{1}{2\times 3}(x-a)^3 + \frac{d^3y}{dx^3}(a)\frac{1}{2}(x-a)^2 + \frac{d^2y}{dx^2}(a)x + C$$

Again solve for C:

$$\frac{dy}{dx}(a) = \frac{d^4y}{dx^4}(a)\frac{1}{2\times 3}(a-a)^3 + \frac{d^3y}{dx^3}(a)\frac{1}{2}(a-a)^2 + \frac{d^2y}{dx^2}(a)a + C$$

$$\frac{dy}{dx}(a) = \frac{d^2y}{dx^2}(a)a + C$$

$$C = \frac{dy}{dx}(a) - \frac{d^2y}{dx^2}(a)a$$

$$\frac{dy}{dx}(x) = \frac{d^4y}{dx^4}(a)\frac{1}{2\times 3}(x-a)^3 + \frac{d^3y}{dx^3}(a)\frac{1}{2}(x-a)^2 + \frac{d^2y}{dx^2}(a)x + \frac{dy}{dx}(a) - \frac{d^2y}{dx^2}(a)a$$

$$\frac{dy}{dx}(x) = \frac{d^4y}{dx^4}(a)\frac{1}{2\times 3}(x-a)^3 + \frac{d^3y}{dx^3}(a)\frac{1}{2}(x-a)^2 + \frac{d^2y}{dx^2}(a)(x-a) + \frac{dy}{dx}(a)$$

Then we try to obtain $y(x)$:

$$y(x) = \frac{d^4y}{dx^4}(a)\frac{1}{2\times 3\times 4}(x-a)^4 + \frac{d^3y}{dx^3}(a)\frac{1}{2\times 3}(x-a)^3 + \frac{d^2y}{dx^2}(a)\frac{1}{2}(x-a)^2 + \frac{dy}{dx}(a)x + C$$

Solve for C:

$$y(a) = \frac{d^4y}{dx^4}(a)\frac{1}{2\times 3\times 4}(a-a)^4 + \frac{d^3y}{dx^3}(a)\frac{1}{2\times 3}(a-a)^3 + \frac{d^2y}{dx^2}(a)\frac{1}{2}(a-a)^2 + \frac{dy}{dx}(a)a + C$$

$$y(a) = \frac{dy}{dx}(a)a + C$$

$$C = y(a) - \frac{dy}{dx}(a)a$$

So that

$$y(x) = \frac{d^4y}{dx^4}(a)\frac{1}{2 \times 3 \times 4}(x-a)^4 + \frac{d^3y}{dx^3}(a)\frac{1}{2 \times 3}(x-a)^3 + \frac{d^2y}{dx^2}(a)\frac{1}{2}(x-a)^2 + \frac{dy}{dx}(a)x + y(a) - \frac{dy}{dx}(a)a$$

$$y(x) = \frac{d^4y}{dx^4}(a)\frac{1}{2 \times 3 \times 4}(x-a)^4 + \frac{d^3y}{dx^3}(a)\frac{1}{2 \times 3}(x-a)^3 + \frac{d^2y}{dx^2}(a)\frac{1}{2}(x-a)^2 + \frac{dy}{dx}(a)(x-a) + y(a)$$

For a general derivation we start with the assumption that the n'th derivative is a constant so that

$$\frac{d^ny}{dx^n}(x) = \frac{d^ny}{dx^n}(a)$$

We can obtain

$$\frac{d^{n-1}y}{dx^{n-1}}(x) = \frac{d^ny}{dx^n}(a)x + C$$

Solve for C by letting x=a

$$\frac{d^{n-1}y}{dx^{n-1}}(a) = \frac{d^ny}{dx^n}(a)a + C$$

$$C = \frac{d^{n-1}y}{dx^{n-1}}(a) - \frac{d^ny}{dx^n}(a)a$$

$$\frac{d^{n-1}y}{dx^{n-1}}(x) = \frac{d^ny}{dx^n}(a)(x-a) + \frac{d^{n-1}y}{dx^{n-1}}(a)$$

And then

$$\frac{d^{n-2}y}{dx^{n-2}}(x) = \frac{d^n y}{dx^n}(a)\frac{1}{2}(x-a)^2 + \frac{d^{n-1}y}{dx^{n-1}}(a)x + C$$

Obtaining C:

$$\frac{d^{n-2}y}{dx^{n-2}}(a) = \frac{d^n y}{dx^n}(a)\frac{1}{2}(a-a)^2 + \frac{d^{n-1}y}{dx^{n-1}}(a)a + C$$

$$C = \frac{d^{n-2}y}{dx^{n-2}}(a) - \frac{d^{n-1}y}{dx^{n-1}}(a)a$$

$$\begin{aligned}\frac{d^{n-2}y}{dx^{n-2}}(x) &= \frac{d^n y}{dx^n}(a)\frac{1}{2}(x-a)^2 + \frac{d^{n-1}y}{dx^{n-1}}(a)x + \frac{d^{n-2}y}{dx^{n-2}}(a) - \frac{d^{n-1}y}{dx^{n-1}}(a)a \\ &= \frac{d^n y}{dx^n}(a)\frac{1}{2}(x-a)^2 + \frac{d^{n-1}y}{dx^{n-1}}(a)(x-a) + \frac{d^{n-2}y}{dx^{n-2}}(a)\end{aligned}$$

From the derivation above we see that

$$C_{\frac{d^k y}{dx^k}(x)} = \frac{d^k y}{dx^k}(a) - \frac{d^{k+1}y}{dx^{k+1}}(a)a$$

We also see a general pattern that for every time we go up to a k'th derivative we always add

$$\frac{d^{k+1}y}{dx^{k+1}}(a)x$$

And that we from the k+1 derivative always have from C

$$\frac{d^{k+1}y}{dx^{k+1}}(a)a$$

So that we always can add

$$\frac{d^{k+1}y}{dx^{k+1}}(a)(x-a)$$

This will for the k-1 derivative become

$$\frac{d^{k+1}y}{dx^{k+1}}(a)\frac{1}{2}(x-a)^2$$

And for the k-2 derivative this will become

$$\frac{d^{k+1}y}{dx^{k+1}}(a)\frac{1}{2\times 3}(x-a)^3$$

Since this pattern is accumulating and we get a new $\frac{d^k y}{dx^k}(a)(x-a)$ for every integration and keep them by integrating them once each time we go from k to $k-1$ derivative we can if the n'th derivative is constant start with

$$\frac{d^n y}{dx^n}(x)=\frac{d^n y}{dx^n}(a)$$

And end up with

$$P_n(x)=y(x)=\frac{d^n y}{dx^n}(a)\frac{1}{2\times 3\times \dots \times n}(x-a)^n+\dots+\frac{d^3 y}{dx^3}(a)\frac{1}{2\times 3}(x-a)^3 + \frac{d^2 y}{dx^2}(a)\frac{1}{2}(x-a)^2+\frac{dy}{dx}(a)(x-a)+y(a)$$

Error Taylor polynomial

A general taylor polynomial where $\frac{d^n y}{dx^n}(x)\neq constant$ and we can't derive it the way we did above has the same formula as above

$$P(x)=\frac{d^n y}{dx^n}(a)\frac{1}{2\times 3\times \dots \times n}(x-a)^n+\dots+\frac{d^3 y}{dx^3}(a)\frac{1}{2\times 3}(x-a)^3+\frac{d^2 y}{dx^2}(a)\frac{1}{2}(x-a)^2 + \frac{dy}{dx}(a)(x-a)+y(a)$$

So for x=a the Taylor polynomial becomes

$$P(a)=y(a)$$

So in a the difference between the Taylor polynomial and the function the taylor polynomial is approximating becomes

$$y(a)-P(a)=y(a)-y(a)=0$$

The first derivative of the taylor polynomial

$$P'(x) = \frac{d^n y}{dx^n}(a)\frac{1}{2 \times 3 \times \ldots \times n} n(x-a)^{n-1} + \cdots + \frac{d^3 y}{dx^3}(a)\frac{1}{2 \times 3} 3(x-a)^{3-1} + \frac{d^2 y}{dx^2}(a)\frac{1}{2} 2(x-a) + \frac{dy}{dx}(a)$$

So we see that

$$y'(a) - P'(a) = \frac{dy}{dx}(a) - \frac{dy}{dx}(a) = 0$$

We also observe that there is a general pattern that gives

$$\frac{d^k y}{dx^k}(a) = \frac{d^k P(a)}{dx^k}$$

So that

$$\frac{d^k y}{dx^k}(a) - \frac{d^k P(a)}{dx^k} = 0$$

All the way up until

$$\frac{d^n y}{dx^n}(a) - \frac{d^n P(a)}{dx^n} = 0$$

The error of the taylor polynomial is

$$E(x) = y(x) - P(x)$$

If we take the n+1 derivative of an n'th degree taylor polynomial it is 0 so that

$$\frac{d^{n+1}E(x)}{dx^{n+1}} = \frac{d^{n+1}y(x)}{dx^{n+1}} - \frac{d^{n+1}P(x)}{dx^{n+1}} = \frac{d^{n+1}y(x)}{dx^{n+1}} - 0 = \frac{d^{n+1}y(x)}{dx^{n+1}} \le M$$

We choose M as the max value of $\frac{d^{n+1}E(x)}{dx^{n+1}}$ on the interval $[a, b]$

$$\frac{d^{n+1}E(c)}{dx^{n+1}} = M$$

Where M is a constant. If we try to integrate this once we would obtain

$$\int \left|\frac{d^{n+1}E(x)}{dx^{n+1}}\right| dx \le \int M\, dx$$

Since $\left|\frac{d^{n+1}E(x)}{dx^{n+1}}\right| > 0$ for all x and $\frac{d^{n+1}E(x)}{dx^{n+1}}$ can be below or above 0 for different x

$$\left|\int \frac{d^{n+1}E(x)}{dx^{n+1}} dx\right| \le \int \left|\frac{d^{n+1}E(x)}{dx^{n+1}}\right| dx$$

With $\left|\int \frac{d^{n+1}E(x)}{dx^{n+1}}dx\right|$ we can obtain $\left|\frac{d^nE(x)}{dx^n}\right|$.

$$\left|\int \frac{d^{n+1}E(x)}{dx^{n+1}}dx\right| \leq \int M\,dx$$

$$\left|\frac{d^nE(x)}{dx^n}\right| \leq Mx + C$$

In order to find the constant C we know that $\frac{d^nE(a)}{dx^n} = 0$

$$\left|\frac{d^nE(a)}{dx^n}\right| = 0 \leq Ma + C$$

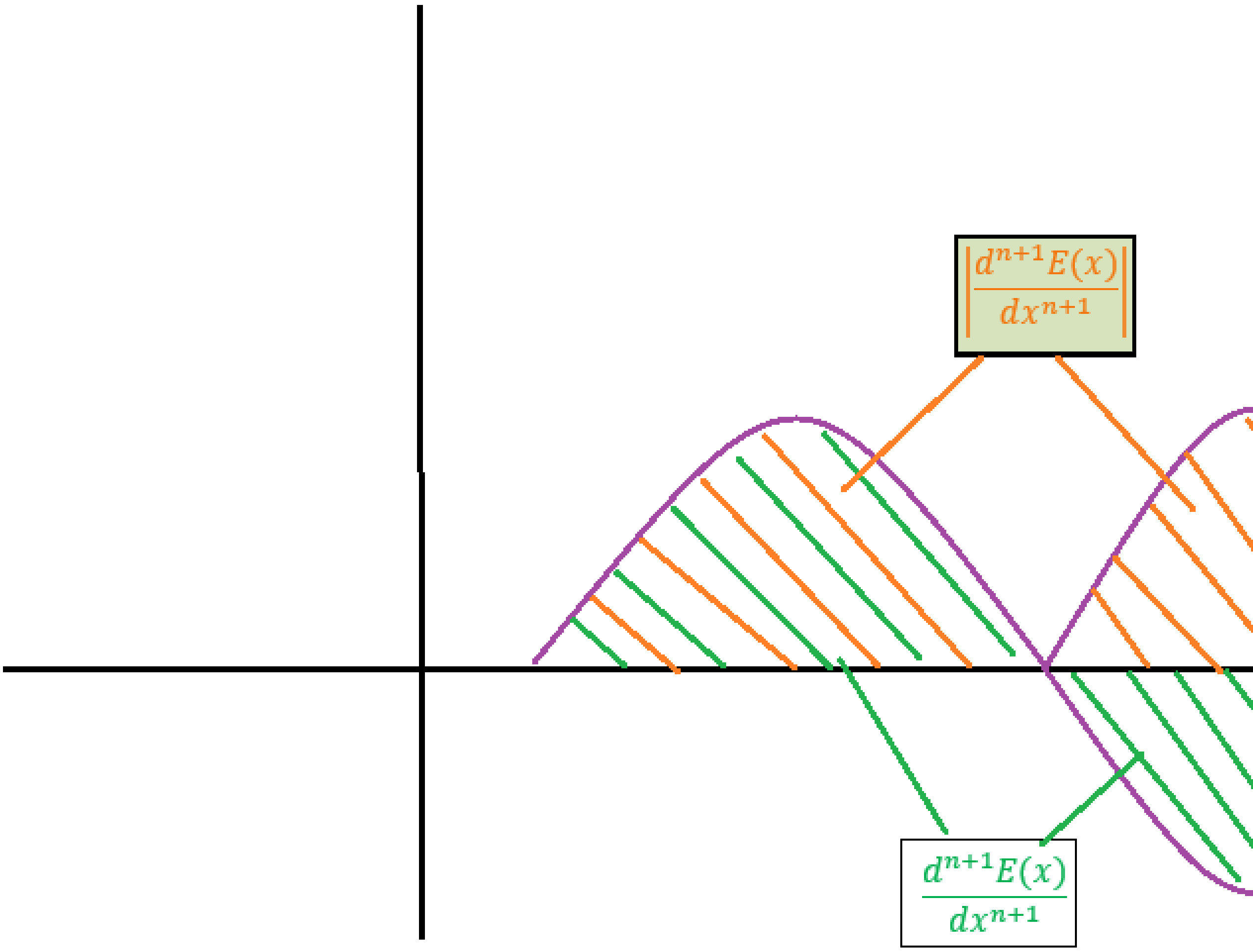
$$-Ma \leq C$$

We could have integrated

$$\int \left|\frac{d^{n+1}E(x)}{dx^{n+1}}\right| dx = \int M\,dx$$

Then we would have obtained that when x=a that this is 0

Since we always have that $\left|\int \frac{d^{n+1}E(x)}{dx^{n+1}}dx\right| \leq \int \left|\frac{d^{n+1}E(x)}{dx^{n+1}}\right| dx$ we also have that $\left|\int \frac{d^{n+1}E(x)}{dx^{n+1}}dx\right|$ is 0 when x=a. This is illustrated by this graph



You could say that $\left|\frac{d^{n+1}E(x)}{dx^{n+1}}\right| \times 1 = M$ is always larger then $\frac{d^{n+1}E(x)}{dx^{n+1}} \times 1$. We are interested in $\frac{d^{n}E(x)}{dx^{n}}$ which is the integral. We know that $\frac{d^{n}E(x)}{dx^{n}}$ is 0 in x=a. We can use this and by using M=$\frac{d^{n+1}E(c)}{dx^{n+1}}$ we reassure that we get a larger value for the integral $\left|\int \frac{d^{n+1}E(x)}{dx^{n+1}} dx\right| \leq \int M\, dx$ since M is based on a function $\left|\frac{d^{n+1}E(x)}{dx^{n+1}}\right| \geq \frac{d^{n+1}E(x)}{dx^{n+1}}$.

$$-Ma = C$$

$$\left|\frac{d^{n}E(x)}{dx^{n}}\right| \leq Mx - Ma = M(x-a)$$

By doing the same steps as above we obtain

$$\left|\frac{d^{n-1}E(x)}{dx^{n}}\right| \leq M\frac{1}{2}(x-a)^2 + C$$

$$0 \leq M\frac{1}{2}(a-a)^2 + C \qquad 0 \leq C$$

Again we solve for C by using the info at x=a which we can use since the same quantification of M as above

$$C = 0$$

$$\left|\frac{d^{n-1}E(x)}{dx^n}\right| \leq M\frac{1}{2}(x-a)^2$$

We repeat the same process until we obtain

$$|E(x)| \leq M\frac{1}{(n+1)!}(x-a)^{2+n-1} = M\frac{1}{(n+1)!}(x-a)^{1+n} = \frac{d^{n+1}E(c)}{dx^{n+1}}\frac{1}{(n+1)!}(x-a)^{1+n}$$